

# DUALITY AND OPERATOR ALGEBRAS II: OPERATOR ALGEBRAS AS BANACH ALGEBRAS

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ABSTRACT. We answer, by counterexample, several open questions concerning algebras of operators on a Hilbert space. The answers add further weight to the thesis that, for many purposes, such algebras ought to be studied in the framework of operator spaces, as opposed to that of Banach spaces and Banach algebras. In particular, the ‘nonselfadjoint analogue’ of a  $W^*$ -algebra resides naturally in the category of dual operator spaces, as opposed to dual Banach spaces. We also show that an automatic  $w^*$ -continuity result in the preceding paper of the authors, is sharp.

## 1. INTRODUCTION

An *operator algebra* is an algebra of operators on a Hilbert space. Since the advent of operator space theory, there has been much progress toward the development of a *general* theory of such algebras (e.g. see [2]). From the ‘operator space perspective’, on an operator algebra  $A$  one should consider not only the norm on  $A$ , but also the canonical norms on the spaces  $M_n(A)$  of matrices with entries in  $A$ , for all  $n \geq 1$ . The obvious question is, is this really necessary for the study of such algebras? While admittedly this is not a well-posed question, since it depends on the applications one has in mind, a recent survey [1] collected some test questions which have resisted solution to date, and whose answers might ‘tip the balance’ on this issue, in some sense. We are now able to answer several of these questions. Our main result may be summarized as saying that Sakai’s famous characterization of von Neumann algebras in terms of  $C^*$ -algebras with a Banach space predual (see [8, Theorem 1.16.7]), is not valid for general operator algebras without using the operator space framework. In particular, we exhibit here an operator algebra with an identity of norm 1, even a commutative one, which has a Banach space predual, but is not homomorphic, via a homeomorphism for the associated weak\* topologies, to any  $\sigma$ -weakly closed (that is, weak\* closed) operator algebra. This rules out the possibility, which had remained open, of a ‘non-operator-space variant’ of the following theorem attributable to Le Merdy and the two authors (e.g. see [2, Section 2.7] and [6, 3]): namely that the  $\sigma$ -weakly closed operator algebras ‘are precisely’ the operator algebras which possess an operator space predual. Thus, we are able to bring to its final form the topic of abstract characterizations of  $\sigma$ -weakly closed operator algebras. We also use our counterexample to deduce that several other known results about operator algebras and operator spaces are not valid if one drops hypotheses involving ‘matrix norms’. For example, we exhibit a subspace  $X$  of a  $C^*$ -algebra  $A$ , and an  $a \in A$  with  $aX \subset X$ , such that  $X$  is isometric to

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a dual Banach space, but the function  $x \mapsto ax$  on  $X$  is not weak\* continuous (we showed in [3] that this function is always weak\* continuous if also  $a^*X \subset X$ , or if  $X$  is a dual operator space). As another byproduct, we have found more simple examples of operator spaces which have a Banach space predual which is not an operator space predual (see [3] for more discussion of this point).

We refer the reader to any of the recently available texts on operator spaces, for more information on that topic if needed. For the duality of operator spaces, we recommend [2, Section 1.4], although this is not necessary for reading our paper. We abbreviate ‘weak\*’ to ‘w\*-’ throughout.

## 2. DUALITY AND LOWERSEMICONTINUITY

To construct ‘noncanonical’ Banach space preduals, the following well known result is very useful. A function defined on a dual Banach space will be called *lower w\*-continuous* if it is lowersemicontinuous with respect to the w\*-topology.

**Lemma 2.1.** ([4, Lemma 3.1], [5, Lemma 8.8]) *If  $X$  is a dual Banach space, and if  $||| \cdot |||$  is an equivalent norm on  $X$ , then the following are equivalent:*

- (i)  $\{x \in X : |||x||| \leq 1\}$  is w\*-closed;
- (ii)  $||| \cdot |||$  is lower w\*-continuous;
- (iii)  $(X, ||| \cdot |||)$  is a dual Banach space, and the associated w\*-topology is the same as the original w\*-topology on  $X$ .

Let  $Y$  be an operator space which is also a non-reflexive dual Banach space, with predual  $Y_*$ . In fact, in our examples,  $Y$  will be a subspace of a unital  $C^*$ -algebra  $A$  with  $1_A \in Y$ . We will identify  $\mathbb{C}$  with  $\mathbb{C}1_A \subset A$ . Suppose that  $T$  is a bounded operator on  $Y$ , which is discontinuous with respect to the w\*-topology of  $Y$ . We then consider operator spaces which may be ‘built’ from  $Y$  and  $T$ . In particular, in the remainder of the paper we will be using the the following four subspaces of  $M_2(A)$ :

$$\begin{aligned} B &= \left\{ \begin{bmatrix} 0 & y \\ 0 & Ty \end{bmatrix}, y \in Y \right\}, \\ C &= \left\{ \begin{bmatrix} a & y \\ 0 & a + Ty \end{bmatrix}, y \in Y, a \in \mathbb{C}1_A \right\}, \\ E &= \left\{ \begin{bmatrix} Ty & y \\ 0 & Ty \end{bmatrix}, y \in Y \right\}, \end{aligned}$$

and

$$D = \left\{ \begin{bmatrix} a + Ty & y \\ 0 & a + Ty \end{bmatrix}, y \in Y, a \in \mathbb{C}1_A \right\}.$$

Notice that  $B, C$ , and  $D$  are operator algebras, subalgebras of  $M_2(A)$ , if  $T$  takes values in  $\mathbb{C}1_A$ . The norms of the matrices in the expressions for  $B$  and  $E$  above, are equivalent to the original norm on  $Y$ , whereas the norms in the expressions for  $C$  and  $D$  are equivalent to the  $\infty$ -direct sum norm on  $Y \oplus \mathbb{C}$ . Our strategy will be as follows. Consider for example the canonical isomorphism  $\theta : Y \oplus^\infty \mathbb{C} \rightarrow D$  given by

$$\theta((y, a)) = \begin{bmatrix} a + Ty & y \\ 0 & a + Ty \end{bmatrix}, \quad y \in Y, a \in \mathbb{C}.$$

The  $M_2(A)$ -norm on  $D$  transfers, via  $\theta$ , to a norm  $||| \cdot |||$  on the dual space  $Y \oplus^\infty \mathbb{C}$ . By Lemma 2.1,  $D$  is a dual Banach space if  $||| \cdot ||| = \|\theta(\cdot)\|_D$  is lower w\*-continuous

with respect to the  $w^*$ -topology induced by the predual  $Y_* \oplus^1 \mathbb{C}$ . Similar (but simpler) considerations apply to the other spaces  $B, C, E$  above. Thus  $B$  is a dual Banach space and the canonical map  $\rho : Y \rightarrow B$  is a  $w^*$ -homeomorphism, if  $\|\rho(\cdot)\|_B$  is lower  $w^*$ -continuous on  $Y$ .

For computing the norms above, we use the following explicit formula:

$$(2.1) \quad \left\| \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \right\|^2 = \frac{1}{2} [|a|^2 + \|x\|^2 + |b|^2 + \sqrt{(|a|^2 + \|x\|^2 + |b|^2)^2 - 4|a|^2|b|^2}],$$

for  $x \in A$  and  $a, b \in \mathbb{C} 1_A$ . This follows easily from the last formula in the proof of 2.2.11 in [2].

Because the computations are quite manageable here, we will henceforth restrict our attention to the case when  $Y = \ell^1$  and  $Y_* = c_0$ . We write  $(e_k)$  for the canonical basis. We will fix an operator space structure on  $Y$  so that the identity of the containing  $C^*$ -algebra  $A$  is  $e_1$ , which we will write as  $1_Y$ . For example, let  $A$  be the commutative  $C^*$ -algebra  $C(\mathbb{T}^\infty)$ , where  $\mathbb{T}$  is the unit circle, and  $Y$  the copy of  $\ell^1$  in  $A$  corresponding to the functions  $(z_k) \mapsto a_1 + \sum_{k \geq 2} a_k z_k$  on  $\mathbb{T}^\infty$ , for all  $(a_k) \in \ell^1$ . (See also e.g. [7, 9.6] or [2, 4.3.8].) Note that  $w^*$ -convergence of bounded nets in  $\ell^1$  simply means ‘component-wise convergence’. We will take the map  $T$  above to be of the form  $Ty = \tau(y)1$ , where  $\tau \in Y^* \setminus Y_*$  is of norm 1. We will need the following:

**Lemma 2.2.** *For a functional  $\tau = (a_j) \in \ell^\infty = (\ell^1)^*$ , the function  $f(y) = \|y\|^2 + |\tau(y)|^2$  is lower  $w^*$ -continuous on  $\ell^1$  (regarded as the dual of  $c_0$ ) if and only if*

$$(2.2) \quad MS \leq 1, \text{ where } M = \sup_j |a_j| \text{ and } S = \limsup_j |a_j|.$$

*Proof.* If  $f$  is lower  $w^*$ -continuous, consider the sequence  $y(m) = ze_1 + we_m$ , where  $z, w \in \mathbb{C}$ . This sequence converges in the  $w^*$ -topology to  $y = ze_1$ . Lower  $w^*$ -continuity then demands that  $f(y) \leq \liminf_m f(y(m))$ . This can be rewritten as

$$(2.3) \quad 0 \leq \liminf_m [(|a_m|^2 + 1)|w|^2 + 2(|z||w| + \operatorname{Re}(a_1 \bar{a}_m z \bar{w}))].$$

If  $a$  is any limit point of the sequence  $(a_m)$ , then by choosing first a subsequence converging to  $a$ , and then appropriate phases of  $z$  and  $w$ , it follows from (2.3) that  $0 \leq (|a|^2 + 1)t^2 + 2st(1 - |a_1||a|)$  for all  $s, t > 0$ . Letting  $t \rightarrow 0$ , we have  $1 - |a_1||a| \geq 0$ . In particular  $|a_1|S \leq 1$ . Similarly,  $|a_k|S \leq 1$  for all  $k$ . Hence  $MS \leq 1$ .

Conversely, note that  $f(y) = |\sum_{j=1}^\infty a_j y_j|^2 + (\sum_{j=1}^\infty |y_j|)^2$  for  $y = (y_j) \in \ell^1$ , so

$$(2.4) \quad f(y) = \sum_{j=1}^\infty (1 + |a_j|^2) |y_j|^2 + 2 \sum_{i < j} (|y_i||y_j| + \operatorname{Re}(a_i \bar{a}_j y_i \bar{y}_j)).$$

Of course,

$$|y_i||y_j| + \operatorname{Re}(a_i \bar{a}_j y_i \bar{y}_j) \geq |y_i||y_j|(1 - |a_i||a_j|).$$

We claim that if  $MS < 1$ , then  $|a_i||a_j| > 1$  only for finitely many pairs  $(i, j)$ , so that almost all the terms on the right side of (2.4) are non-negative. Indeed, if this were not true, then there are two possibilities: (1) there exists an  $i$  such that  $|a_i||a_j| > 1$  for infinitely many  $j$ 's, or (2) there exist infinitely many  $i$ 's such that for some  $j(i)$  we have  $|a_i||a_{j(i)}| > 1$ . In the first case, it follows that  $|a_i|S \geq 1$ , hence  $MS \geq 1$ , which contradicts the assumption. Similarly, in the second case we have  $|a_i|M > 1$ , hence  $SM \geq 1$ , again a contradiction.

Thus if  $MS < 1$ , the sum  $\sum_{i < j} |y_i| |y_j| + \operatorname{Re}(a_i \bar{a}_j y_i \bar{y}_j)$  splits into a finite partial sum, and an infinite sum in which all the terms  $|y_i| |y_j| + \operatorname{Re}(a_i \bar{a}_j y_i \bar{y}_j)$  are non-negative. The finite partial sum is actually  $w^*$ -continuous. The other sum is the supremum of its own finite partial sums, each of which is  $w^*$ -continuous. Since a supremum of lowersemicontinuous functions is lowersemicontinuous, this proves that our infinite sum is lower  $w^*$ -continuous. A similar but easier argument shows that  $\sum_{j=1}^{\infty} (1 + |a_j|^2) |y_j|^2$  is lower  $w^*$ -continuous.

If  $MS = 1$ , we note that  $f$  is the supremum of functions  $f_n(y) = \|y\|^2 + |\tau_n(y)|^2$ , where  $\tau_n = (1 - \frac{1}{n})\tau$  ( $n = 1, 2, \dots$ ). Since the functions  $f_n$  are lower  $w^*$ -continuous by what we have already proved, so must be  $f$ .  $\square$

### 3. CONSEQUENCES

**Corollary 3.1.** *There exists an operator algebra  $B$ , which is a dual Banach space, and an idempotent element  $p \in B$ , such that  $\dim(pB) = 1$ , and such that left multiplication by  $p$  is not  $w^*$ -continuous on  $B$ .*

*Proof.* We use the notation in the previous section. Choose  $\tau$  to satisfy the conditions of Lemma 2.2, and  $\tau(1_Y) = a_1 = 1$ . Let  $T = \tau(\cdot)1_Y$ , and let  $B$  be as in the last section, with  $\rho : Y \rightarrow B$  the canonical map. The  $M_2(A)$ -norm on  $B$  is given by the expression  $\|\rho(y)\|_B = \sqrt{\|y\|^2 + |\tau(y)|^2}$  for  $y \in Y$  (by e.g. (2.1)). By Lemma 2.2, this quantity is lower  $w^*$ -continuous. It follows, as in the arguments a couple of paragraphs above Lemma 2.2, that  $B$  is a dual Banach space, with  $w^*$ -topology determined by the canonical pairing with  $c_0$ . If  $p \in B$  corresponds to  $y = e_1 = 1_Y \in Y = \ell^1$ , then  $p$  is idempotent,  $pB$  is one-dimensional, and the map  $b \rightarrow pb$  on  $B$  corresponds to the map  $T$  on  $Y$ , which is not  $w^*$ -continuous.  $\square$

Such an algebra  $B$  is not isomorphic (in the appropriate sense) to a  $\sigma$ -weakly closed operator algebra, since the product on any  $\sigma$ -weakly closed operator algebra is separately  $w^*$ -continuous. To obtain a ‘unital’ counterexample is a little harder:

**Theorem 3.2.** *There exists a commutative operator algebra  $D$  with an identity of norm 1, which is a dual Banach space, and a nilpotent element  $p \in D$ , such that left multiplication by  $p$  is not  $w^*$ -continuous on  $D$ . Moreover,  $D$  is not homomorphic, via a homeomorphism for the associated weak\* topologies, to any  $\sigma$ -weakly closed operator algebra.*

*Proof.* The last assertion here follows as in the line above the theorem.

We employ a similar strategy to that of Corollary 3.1, and the notation in the previous section. Set  $\tau_0 = (0, 1, 1, 1, \dots) \in \ell^\infty$ ,  $\tau = \frac{1}{4}\tau_0$ , and  $T = \tau(\cdot)1_Y$ . Consider the unital operator algebra  $D$  above, which in this case consists of all matrices

$$(3.1) \quad x = \begin{bmatrix} b & y \\ 0 & b \end{bmatrix}, \text{ where } y \in Y \text{ and } b \in \mathbb{C}.$$

Clearly  $D$  contains the space  $E$  as a subspace. Let  $p \in E$  correspond to  $y = e_1 = 1_Y \in Y = \ell^1$ . We claim that it suffices to prove that the unit ball for the norm  $\|\cdot\|$  on  $Y \oplus \mathbb{C}$ , defined a couple of paragraphs above Lemma 2.2, is  $w^*$ -closed in the topology given by the canonical pairing with  $c_0 \oplus \mathbb{C}$ . (This corresponds to the pairing of a matrix  $x$  of the form (3.1) and an element  $v = (z, \beta) \in c_0 \oplus \mathbb{C}$ , via the formula  $\langle x, v \rangle = \langle y, z \rangle_{\ell^1, c_0} + \beta(b - \tau(y))$ .) If this is the case, then  $D$  is a dual Banach space by Lemma 2.1, and  $E$  is a  $w^*$ -closed subspace, but left multiplication by  $p$  is not  $w^*$ -continuous on  $E$ , and hence not on  $D$ , since  $\tau \notin c_0$ .

By (2.1), the norm of a matrix of the form (3.1) is  $\|x\|^2 = \frac{1}{2}[2|b|^2 + \|y\|^2 + \sqrt{\|y\|^4 + 4|b|^2\|y\|^2}]$ . From this (or otherwise) it follows by elementary algebraic manipulations that

$$(3.2) \quad \|x\| \leq 1 \iff \|y\| + |b|^2 \leq 1.$$

In the topology described a couple of paragraphs above Lemma 2.2, the convergence of nets in  $D$  is as follows:

$$\begin{bmatrix} a_t + Ty_t & y_t \\ 0 & a_t + Ty_t \end{bmatrix} \rightarrow \begin{bmatrix} a + Ty & y \\ 0 & a + Ty \end{bmatrix} \iff a_t \rightarrow a \text{ in } \mathbb{C}, y_t \rightarrow y \text{ w}^* \text{ in } Y.$$

To prove that the unit ball of  $D$  is closed in this topology, by (3.2) it suffices to show that the function

$$\begin{bmatrix} a + Ty & y \\ 0 & a + Ty \end{bmatrix} \mapsto \|y\| + |a + \tau(y)|^2$$

is lower  $w^*$ -continuous on  $\text{Ball}(D)$ , in the latter topology. Since  $\text{Ball}(D)$  is bounded and  $a \in \mathbb{C}$ , the proof reduces to showing that for a fixed  $a \in \mathbb{C}$ , the function

$$g(y) = \|y\| + |a + \tau(y)|^2 = |a|^2 + \|y\| + \frac{1}{2}\text{Re}(\bar{a}\tau_0(y)) + \frac{1}{16}|\tau_0(y)|^2$$

is lower  $w^*$ -continuous on  $Y$ . We can rewrite  $g(y)$  in the form

$$|a|^2 + \left( \frac{7}{8}\|y\| + \frac{1}{2}\text{Re}(\bar{a}\tau_0(y)) \right) + \frac{1}{4} \left( \frac{1}{2}\|y\| - \frac{1}{4}\|y\|^2 \right) + \frac{1}{16} (\|y\|^2 + |\tau_0(y)|^2).$$

The last term in this expression is lower  $w^*$ -continuous by Lemma 2.2. Since the norm is lower  $w^*$ -continuous and the function  $t \mapsto t/2 - t^2/4$  is increasing on the interval  $[0, 1]$ , it follows that the term  $\|y\|/2 - \|y\|^2/4$  is also lower  $w^*$ -continuous. It remains to prove that the second term in the last centered equation is lower  $w^*$ -continuous. But this term can be written as  $1/2$  of

$$\frac{7}{4}\|y\| + \text{Re}(\bar{a}\tau_0(y)) = \frac{7}{4}|y_1| + \sum_{j=2}^{\infty} \left( \frac{7}{4}|y_j| + \text{Re}(\bar{a}y_j) \right).$$

In the last sum all the terms are nonnegative, since  $\frac{7}{4}|y_j| + \text{Re}(\bar{a}y_j) \geq \frac{7}{4}|y_j| - |y_j| \geq 0$ . Hence the lower  $w^*$ -continuity follows, as in the proof of Lemma 2.2, by considering first the finite partial sums.  $\square$

The reader may wonder why, in the last theorem, we did not use the algebra  $C$  from the last section, constructed from the simpler algebra  $B$  used in the proof of Corollary 3.1, by simply adjoining the identity of  $M_2(A)$ . In fact, this construction does not produce a dual Banach space. We may use this observation to show that another result which is valid for operator algebras which possess an operator space predual, fails if we assume only a Banach space predual:

**Corollary 3.3.** *There exists an operator algebra  $A$  which possesses a Banach space predual, such that the unitization  $A^1$  of  $A$  (see [2, Section 2.1]), possesses no Banach space predual for which the embedding of  $A$  in  $A^1$  is  $w^*$ -continuous. In contrast, there can exist no such example which possesses an operator space predual.*

*Proof.* The last assertion follows from the characterization of dual operator algebras mentioned in the first paragraph of our paper, together with [2, Proposition 2.7.4]. For the first assertion, construct  $C$  as mentioned a couple of paragraphs above, using the functional  $\tau = (1, 1, \dots) \in \ell^\infty = (\ell^1)^*$ . Suppose that  $C$  possessed a Banach

space predual for which the canonical embedding of  $B$  in  $C$  was  $w^*$ -continuous. By a variant of the Krein-Smulian theorem,  $B$  would then be  $w^*$ -closed in  $C$ , and the embedding a  $w^*$ -homeomorphism. Let  $\chi : C \rightarrow \mathbb{C}$  be evaluation at the 1-1 entry, a contractive homomorphism. Since the kernel  $B$  of  $\chi$  is  $w^*$ -closed,  $\chi$  is  $w^*$ -continuous. It follows that the map  $c \mapsto (c - \chi(c)1_C, \chi(c))$  is a  $w^*$ -homeomorphism from  $C$  onto  $B \oplus^\infty \mathbb{C}$ . Thus a net  $(b_t + \lambda_t 1_C)$  in  $C$   $w^*$ -converges to  $b + \lambda 1_C$  if and only if  $\lambda_t \rightarrow \lambda$  in  $\mathbb{C}$  and  $b_t \rightarrow b$  in the  $w^*$ -topology in  $B$ . The latter condition simply says that  $y_t \rightarrow y$  in the  $w^*$ -topology of  $\ell^1$ , if  $y_t$  and  $y$  are the corresponding (via  $\rho$ ) elements in  $Y = \ell^1$ . By Lemma 2.1, any closed ball centered at the origin in  $C$  is  $w^*$ -closed with respect to the topology just described on  $C$ . However, if  $y_m = e_1 - \frac{1}{2}e_m$ , and  $b_m$  is the corresponding element of  $B$ , then  $(b_m + 1_C)$  is a net in  $C$  with  $w^*$ -limit  $b + 1_C$ , where  $b \in B$  corresponds to  $e_1 \in Y$ . But one may easily check using (2.1) that  $b + 1_C$  lies outside a ball centered at the origin in  $C$  which contains all the terms  $b_m + 1_C$ .  $\square$

Theorem 3.2 shows that the following is the best result along the lines above, which one can hope for in the ‘Banach algebra category’.

**Proposition 3.4.** *An operator algebra which has a Banach space predual, and whose product is separately  $w^*$ -continuous, is isometric via a homomorphism which is also a  $w^*$ -homeomorphism, to a  $\sigma$ -weakly closed operator algebra.*

*Proof.* This is a remark in the Notes to Section 2.7 in [2]. Indeed, it follows from Le Merdy’s proof in [6].  $\square$

We also have the following positive result in the case of a Banach space predual:

**Proposition 3.5.** *Let  $B$  be a unital operator algebra which is a dual Banach space. Then  $\Delta(B) = B \cap B^*$  is a  $W^*$ -algebra, and if  $b \in \Delta(B)$  then the maps  $a \mapsto ab$  and  $a \mapsto ba$  on  $B$ , are  $w^*$ -continuous.*

*Proof.* Let  $A = B^{**}$  and let  $q : A \rightarrow B$  be the canonical projection (the adjoint of the inclusion  $B_* \subset B^*$ ). Since  $q(1) = 1$ ,  $q$  takes Hermitian elements to Hermitian elements. That is,  $q$  induces a  $w^*$ -continuous projection  $Q$  of  $\Delta(A)$  onto  $\Delta(B)$ . Thus  $\Delta(B)$  is isometric to the dual space  $\Delta(A)/\text{Ker}(Q)$ , and so  $\Delta(B)$  is a  $W^*$ -algebra. The last part follows from e.g. [3, Theorem 3.3].  $\square$

Theorem 3.2 also yields solutions to a couple of other interesting questions, as we discuss next.

A famous theorem of Tomiyama characterizes ‘conditional expectations’ on  $C^*$ -algebras as the contractive projections (e.g. see [8, Remark 2.6.5]). There is a known analogue of this for nonselfadjoint operator algebras, but it applies to *completely contractive* projections (see [2, Corollary 4.2.9]). We are now able to solve the problem of whether contractive projections suffice here. This again illustrates some limitations of the Banach algebra category when studying operator algebras.

**Corollary 3.6.** *There exists a commutative operator algebra  $A$  with an identity of norm 1, a contractive projection  $P$  on  $A$  whose range is a subalgebra  $B$  containing  $1_A$ , and elements  $a \in A, b \in B$ , such that  $P(ba) \neq bP(a)$ .*

*Proof.* If there existed no operator algebra with this property, then it is shown in [1] that Theorem 3.2 would fail.  $\square$

**Remark:** The fact that the algebra in Corollary 3.6 is commutative, also appears to rule out the existence of a ‘Jordan algebra variant’ of Tomiyama’s result for contractive projections, in the setting of nonselfadjoint operator algebras. We thank J. Arazy for discussions on this point in 2002; he also suggested (with a different proof) the following partial result (which is somewhat related to Proposition 3.5): Namely, if  $A$  is an operator algebra with an identity of norm 1, and if  $P$  is a contractive projection on  $A$  whose range is a subalgebra  $B$  containing  $1_A$ , then  $P(ba) = bP(a)$  for all  $a \in A, b \in \Delta(B) = B \cap B^*$ . Indeed this follows from Lemma 3.2 of [3].

Left multiplication by a fixed element of an operator algebra, is an example of an *operator space left multiplier* (e.g. see [3] or [2, Chapter 4] for the full definition of the latter notion). In stark contrast to Theorem 4.1 of [3], which is valid for operator spaces which have an operator space predual, we see:

**Corollary 3.7.** *There exists an operator space  $B$ , which is a dual Banach space, and a left multiplier of  $B$ , which is not  $w^*$ -continuous on  $B$ . In fact such multipliers may be chosen to be idempotent, or nilpotent.*

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